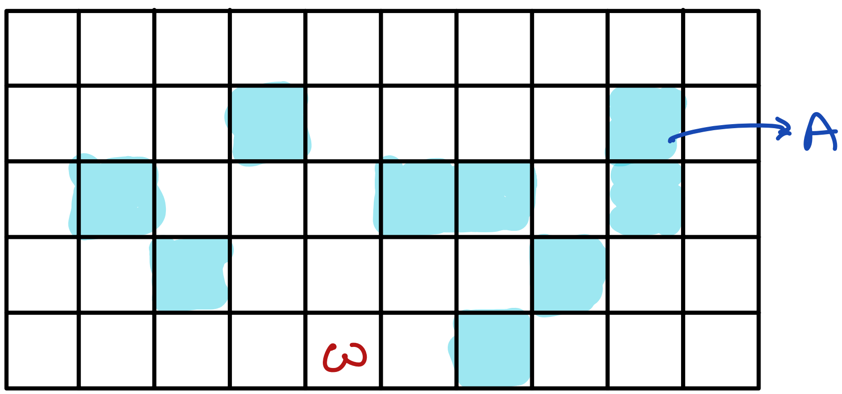
Probability over uncountably infinite spaces

Discrete Case



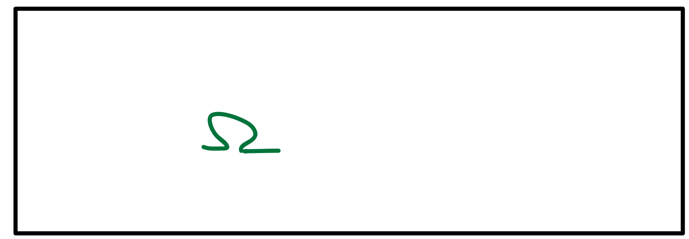
Ω "outcome space"

$\nu : \Omega \longrightarrow [0,1] \quad \sum \nu(\omega) = 1$

Event : $A \subseteq \Omega$

$$\mathbb{P}[A] = \sum_{\omega \in A} \nu(\omega)$$

Uncountably infinite case



$\nu : \Omega \longrightarrow [0,1]$ (say)

$$S_n = \left\{ x \mid \nu(x) \geq \frac{1}{n} \right\} \subseteq \Omega$$

$$\nu(x) > 0 \iff x \in \bigcup_n S_n$$

$$\frac{1}{n} \cdot |S_n| \leq \sum_{x \in S_n} \nu(x) \leq 1$$

$$|S_n| \leq n$$

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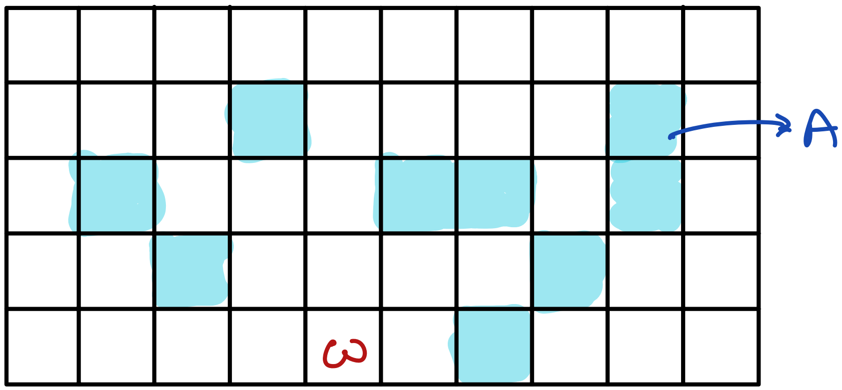

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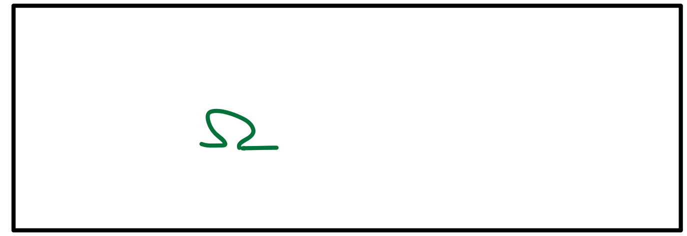

$\nu : \Omega \longrightarrow [0,1]$

$\mathbb{P}[A] = \sum_{\omega \in A} \nu(\omega)$

- Cannot define $\nu : \Omega \longrightarrow [0,1]$
- Only define $\mathbb{P}[A]$ for "allowed events" A

Collections of "allowed events": σ-algebras (σ-fields)

- A collection $\mathcal{F}$ of subsets of Ω is a σ-algebra if

  - $\phi \in \mathcal{F}$
  - $A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$
  - $A_1, A_2 \in \mathcal{F} \Rightarrow A_1 \cup A_2 \in \mathcal{F}$

  (for countable sequence $A_1, A_2, \ldots \in \mathcal{F} \Rightarrow A_1 \cup A_2 \cdots \in \mathcal{F}$)

Ex: $A_1, A_2 \in \mathcal{F} \Rightarrow A_1 \cap A_2 \in \mathcal{F}$

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Prob. Space: $(\Omega, \mathcal{F}, \mu)$

(for countable sequence $A_1, A_2, .. \in \mathcal{F} \Rightarrow A_1 \cup A_2 \dots \in \mathcal{F}$)

Ex: $A_1, A_2 \in \mathcal{F} \Rightarrow A_1 \cap A_2 \in \mathcal{F}$

- Given σ-algebra $\mathcal{F}$, $\nu : \mathcal{F} \to [0,1]$ is a probability measure if

  - $\nu(\phi) = 0 = 1 - \nu(\Omega)$
  - $\nu(A^c) = 1 - \nu(A)$
  - For disjoint $A_1, A_2 \dots \in \mathcal{F}$, $\nu[A_1 \cup A_2 \dots] = \sum_i \nu(A_i)$

Defining the "uniform distribution" over $[0,1]$

$[a,b] \subseteq \Omega$. Would like

$$\nu([a,b]) = b-a$$

$\Omega = [0,1]$

- σ-algebra must include all intervals $[a,b]$

  (Borel σ-algebra. also over $\mathbb{R}$)

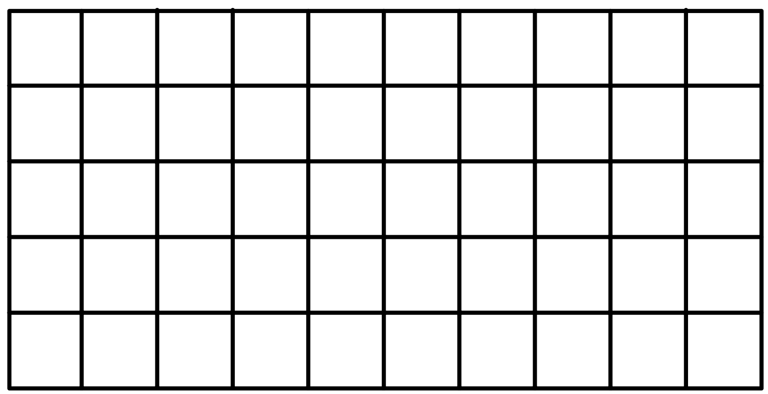
- $\nu : \mathcal{F} \longrightarrow [0,1]$ must satisfy $\nu([a,b]) = b-a$

  (Lebesgue measure)

- Also extends to $[0,1]^d$ $(\mathbb{R}^d)$

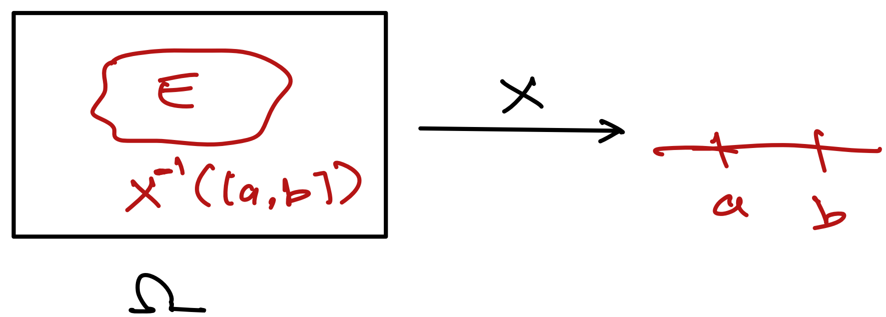
Random Variables

Discrete Case



$X : \Omega \to \mathbb{R}$

Uncountably infinite case

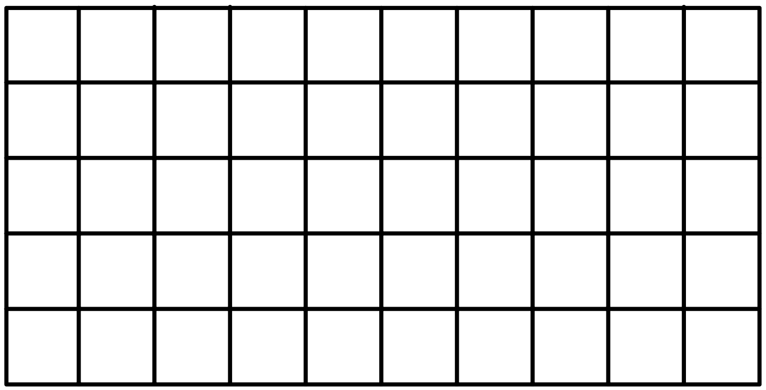


$P[X \in [a,b]] = ?$

$= \mathbb{P}[X^{-1}([a,b])]$

Random Variables

Discrete Case



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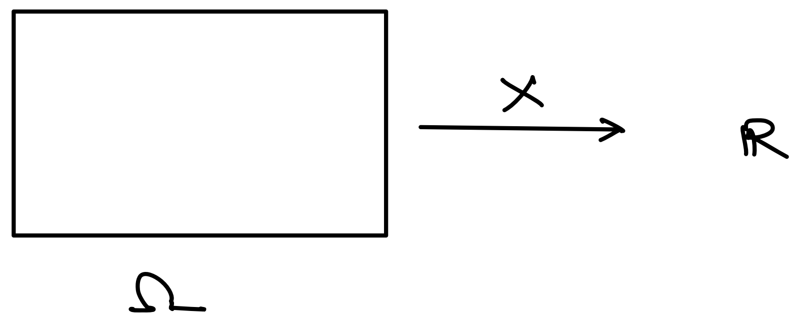


$X : (\Omega, \mathcal{F}) \longrightarrow (\mathbb{R}, \mathcal{B})$

measurable

if $\forall\, A \in \mathcal{B}$, $X^{-1}(A) \in \mathcal{F}$

$\mathbb{P}[X \in A]$ is well-defined

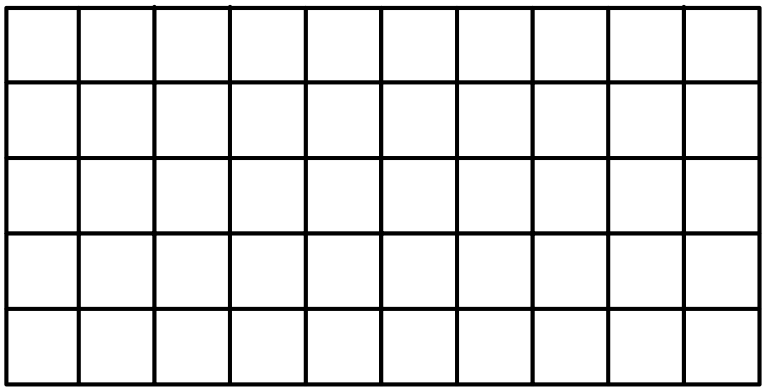
<u>Ex</u>: $X_1, X_2 : (\Omega, \mathcal{F}) \longrightarrow (\mathbb{R}, \mathcal{B})$ measurable

$\Rightarrow c_1 X_1 + c_2 X_2$ measurable

(also $X_1 \cdot X_2$ measurable)

Expectations

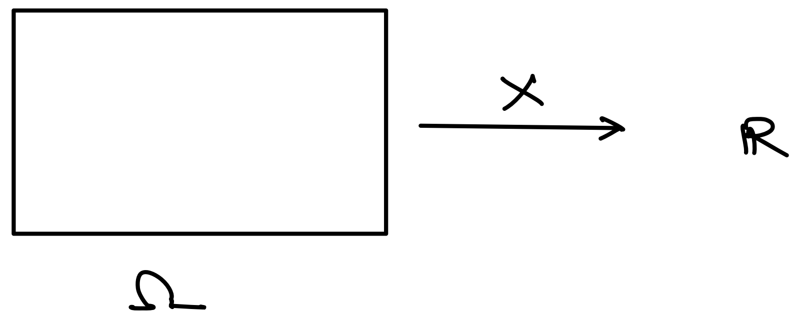
Discrete Case



$$X : \Omega \longrightarrow \mathbb{R}$$

$$\mathbb{E}[X] = \sum \nu(\omega) \cdot X(\omega)$$

Uncountably infinite case



$$X : (\Omega, \mathcal{F}) \longrightarrow (\mathbb{R}, \mathcal{B})$$

measurable

$$\mathbb{E}[X] =$$

Continuous Random Variables

(say) $X : (\Omega, \mathcal{F}) \longrightarrow (\mathbb{R}, \mathcal{B})$

For $a < b$

$$\mathbb{P}[X \in (a, b]) = \mathbb{P}[X \le b] - \mathbb{P}[X \le a] = \int_a^b f(x)\,dx$$

Let $F(x) = \mathbb{P}[X \le x]$

if $\exists$ function $f(x)$ s.t. $F(x) = \int_{-\infty}^{x} f(y)\,dy$

then X is called a continuous random variable

with density function $f(x)$

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} y \cdot f(y) \cdot dy$$

What about all the other stuff

Continuous random variable X with density f

$$\mu = \mathbb{E}[X] = \int_{-\infty}^{\infty} y \cdot f(y) \cdot dy \qquad \mathbb{E}[X^2] = \int_{-\infty}^{\infty} y^2 \cdot f(y) \cdot dy$$

$$\mathrm{Var}[X] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = \mathbb{E}[(X-\mu)^2]$$

Markov : For non-negative r.v. Z

$$\mathbb{E}[Z] = \mathbb{E}[\underbrace{Z \cdot (\mathbb{1}_{\{Z \geq t\}}}_{\text{measurable}} + \mathbb{1}_{\{Z < t\}})]$$

$$\geq \mathbb{E}[Z \cdot \mathbb{1}_{\{Z \geq t\}}] \quad \text{using } Z \geq 0$$

$$\geq \mathbb{E}[t \cdot \mathbb{1}_{\{Z \geq t\}}] \quad \text{using } Z \cdot \mathbb{1}_{\{Z \geq t\}} \geq t \cdot \mathbb{1}_{\{Z \geq t\}}$$

Chebyshev : Markov on $Y = (X-\mu)^2$

Multiple Independent Variables

$\underbrace{X : \Omega \longrightarrow \mathbb{R}}_{\text{density } f(x)}$ $\qquad$ $\underbrace{Y : \Omega \longrightarrow \mathbb{R}}_{\text{density } g(y)}$

$$\mathbb{P}[(X, Y) \in [a,b] \times [c,d]] = \mathbb{P}[X \in [a,b]] \cdot \mathbb{P}[Y \in [c,d]]$$

$$= \int_a^b f(x) \cdot dx \int_c^d g(y)\, dy$$

$$= \int_a^b \int_c^d f(x)\, g(y)\, dx\, dy$$

Density for $(X, Y) : \Omega \longrightarrow \mathbb{R}^2 = f(x)\, g(y)$

Gaussian Random Variables

A specific (very useful!) continuous random variable

$X \sim \underbrace{N(\mu, \sigma^2)}$ density $\gamma_{\mu,\sigma}(x) = \dfrac{e^{-(x-\mu)^2/2\sigma^2}}{\sqrt{2\pi}\,\sigma}$

Normal distribution
with mean μ, variance σ^2

$X \sim N(0,1)$ $\gamma(x) = \dfrac{e^{-x^2/2}}{\sqrt{2\pi}}$

(Standard Normal Distribution)

$$P[X \in [a,b]] = \int_a^b \gamma(x)\,dx = \int_a^b \frac{e^{-x^2/2}}{\sqrt{2\pi}}\,dx$$